(Sequential) topological complexity of aspherical spaces and $\mathcal{A}$-genus

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ADAM MICKIEWICZ UNIVERSITY IN POZNAŃ

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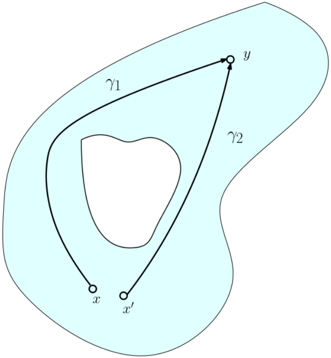

A topological feature of the configuration space inducing instability on the motion planning.

Topological complexity and sectional category

Topological complexity (Farber '01)

$$\mathrm{TC}(X) := \min\{k \in \mathbb{N}_0 \mid \exists \{U_i\}_{0\le i\le k} \text{ open cover of } X\times X \text{ s.t. for every } U_i$$
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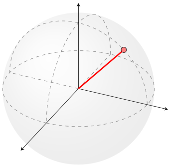
- $\mathrm{secat}(p \colon E \to B)$ equals smallest $k \ge 0$ s.t. $\underbrace{p * p * \cdots * p}_{k} \colon *^k E \to B$ has section (here we mean "fiberwise" join).

Some robots and their complexities!



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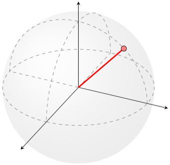

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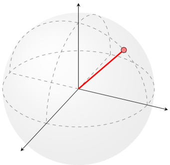

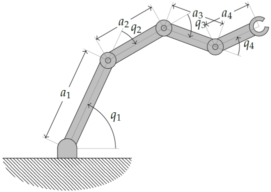

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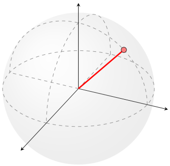

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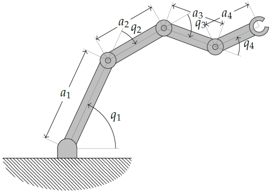

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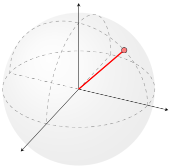

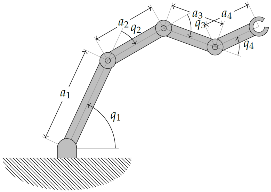

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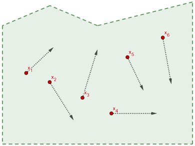

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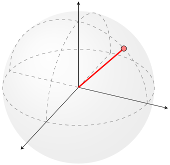

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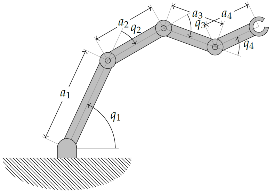

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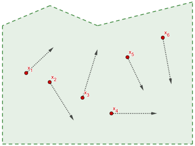

$$\mathrm{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n-2 & \text{for all } m \text{ odd} \\ 2n-3 & \text{for all } m \text{ even} \end{cases}$$

(Farber-Yuzvinsky '04, Farber-Grant '08)

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- It models the motion planning with prescribed $r-2$ intermediate stops between start and end point.
- For $r = 2$ we recover $\mathrm{TC}_2(X) = \mathrm{TC}(X)$.

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An interesting homotopy invariant connected with classic invariants (LS-cat, secat...) with its own open problems like the Eilenberg-Ganea problem.

The Eilenberg-Ganea problem

Henceforth all groups will be discrete.

$$K(G,1) \qquad \begin{cases} \pi_1(K(G,1)) = G \\ \pi_k(K(G,1)) = 0 \ \forall k > 1. \end{cases}$$

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The problem remains open.

Thus, understanding the (sequential) topological complexity of $K(G,1)$-spaces is a prized objective.

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- Farber-Mescher '20: lower bound by dimensions of centralizers.
- Dranishnikov '20: G hyperbolic, $G \not\cong \mathbb{Z} \Rightarrow \mathrm{TC}(G) = 2\mathrm{cd}(G)$.
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For $\mathrm{TC}_r(G)$, $r \geq 2$:

- Basabe-González-Rudyak-Tamaki '14: $\mathrm{TC}_r(\mathbb{Z}^n) = (r-1)\mathrm{cd}(\mathbb{Z}^n) = (r-1)n$.
- Farber-Oprea '19: generalize FGLO bounds.
- Hughes-Li '22: G hyperbolic, $G \ncong \mathbb{Z} \Rightarrow \mathrm{TC}_r(G) = r\mathrm{cd}(G)$.
- EB-Farber-Mescher-Oprea '23 lower bounds for $\mathrm{secat}(H \hookrightarrow G)$ and $\mathrm{TC}_r(\pi)$, with applications to non-aspherical spaces.

Sectional category of subgroup inclusions

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Denote $E_{\mathcal{F}}G$ as the classifying space for the family of subgroups $\mathcal{F}$.

- All isotropy subgroups belong to $\mathcal{F}$.
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Sectional category of subgroup inclusions

For $\iota\colon H \hookrightarrow G$ define the sectional category of the monomorphism ι by

$$\mathrm{secat}(H \hookrightarrow G) := \mathrm{secat}(K(\iota, 1)\colon K(H,1) \to K(G,1))$$

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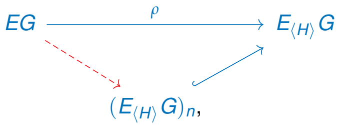
Theorem (Błaszczyk, Carrasquel, EB '20)

$\mathrm{secat}(H \hookrightarrow G)$ *coincides with min.* $n \geq 0$ *s.t.* $\rho\colon EG \to (E_{\langle H\rangle}G)_n$ *can be factorized up to G-homotopy as*



$\mathcal{A}$-genus and its properties

G group, X a G-space, $\mathcal{A}$ a family of G-spaces.

(Bartsch'90)
$\mathcal{A}\text{-genus}(X) := \min\{k \in \mathbb{N}_0 \mid \exists \{U_i\}_{0 \leq i \leq k}$ open cover of X by G-invariant subsets s.t. for every $0 \leq i \leq k$ $\exists A_i \in \mathcal{A}$ and G-equivariant map $U_i \to A_i\}$

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Satisfies

(a) *(Crucial property):* $\mathcal{A}\text{-genus}(X)$ is the smallest integer $k \geq 0$ such that there exists $A_0, \cdots, A_k \in \mathcal{A}$ and a G-equivariant map

$$X \to A_0 * \cdots * A_k.$$

(b) *Monotonicity:* If $\exists$ a G-equivariant map $X \to Y$ then $\mathcal{A}\text{-genus}(X) \leq \mathcal{A}\text{-genus}(Y)$.

(c) *Normalization:* If $A \in \mathcal{A}$ then $\mathcal{A}\text{-genus}(A) = 0$.

(d) $H \leq G$, $\mathcal{A}$ a set of G-spaces and $\mathcal{B}$ a set of H-spaces. For any G-space X

$$\mathcal{B}\text{-genus}(X) \leq (\mathcal{A}\text{-genus}(X) + 1)(\max\{\mathcal{B}\text{-genus}(A) : A \in \mathcal{A}\} + 1) - 1$$

secat of connected covers as $\mathcal{A}$-genus

Theorem (The "main theorem" EB '24)

X path conn. CW-complex. If $q : \widehat{X} \to X$ is a conn. covering, then

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$$\widetilde{X} \to *^{k+1} \left[\left(\pi_1(X) \big/ \pi_1(\widehat{X}) \right) \right].$$

Apply then crucial property.

$\mathrm{secat}(H \hookrightarrow G)$ and TC_r as $\mathcal{A}$-genus

Corollary (EB '24)

G discrete group and $H \leqslant G$. Then $\mathrm{secat}(H \hookrightarrow G) = \mathcal{A}\text{-genus}(EG)$ *where* $\mathcal{A} = \{G/H\}$.

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Let $r \geq 2$, and X be a path conn. CW-complex with $\pi_1(X) = \pi$. Put $\mathcal{A} := \left\{ \pi^r \big/ \Delta_{\pi,r} \right\}$.

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New upper bounds

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If we take $K = \Delta_{\pi,r} \cong \pi$, we recover upper bound from Farber-Oprea '19.

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Theorem (EB'24)

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(a) *For every family $\mathcal{F}$ of subgroups of G we have* $\mathrm{secat}(H \hookrightarrow G) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(G))$.

(b) *For $K \leqslant G$ subconjugate to H s.t.* $\mathrm{cd}_{\langle K \rangle} G \geq 3$ *we have* $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle K \rangle} G$.

(c) *Under the hypothesis of (b), if $K \trianglelefteq G$ then* $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}(G/K)$.

Corollary (EB'24)

π torsion-free group, put $\mathcal{A} = \left\{ \pi^r \big/ \Delta_{\pi,r} \right\}$ and $K \leqslant \pi^r$ subconjugate to $\Delta_{\pi,r}$ s.t. $\mathrm{cd}_{\langle K \rangle} \pi^r \geq 3$.

(a) $\mathrm{TC}_r(\pi) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(\pi^r))$ *for $\mathcal{F}$ any family of subgroups of π.*

(b) $\mathrm{TC}_r(\pi) \leq \mathrm{cd}_{\langle K \rangle} \pi^r$.

(c) $\mathrm{TC}_r(\pi) \leq \mathrm{cd}(\pi^r / K)$ *if $K \trianglelefteq \pi^r$.*

If we take $K = \Delta_{\pi,r} \cong \pi$, we recover upper bound from Farber-Oprea '19.
If $H \leqslant \pi$ central, $\Delta_r(H) \leqslant \pi^r$ is normal. Corollary (c) recovers Grant '12

$$\mathrm{TC}(\pi) \leq \frac{\mathrm{cat}(\pi \times \pi)}{Z(\pi)}$$

for $r = 2$, and generalizes to $r > 2$.

Even more new bounds

Corollary (EB '24)

π torsion-free group, $H, K \leqslant \pi$, and $J \leqslant H$. Then

$$\mathrm{secat}(J \hookrightarrow H) \leq (\mathrm{secat}(K \hookrightarrow \pi) + 1)(\mathcal{B}\text{-genus}((\pi/K)) + 1) - 1$$

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For $\Delta_{H,r} \hookrightarrow H^r$ and $\Delta_{\pi,r} \hookrightarrow \pi^r$ it becomes

$$\mathrm{TC}_r(H) \leq (\mathrm{TC}_r(\pi) + 1)\left(\mathcal{B}\text{-genus}\left(\pi^r / \Delta_{\pi,r}\right) + 1\right) - 1$$

where $\mathcal{B} = \{H^r / \Delta_{H,r}\}$ and $r \in \mathbb{N}$ with $r \geq 2$.

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Let H and K be torsion free groups. Then $\mathrm{TC}_r(H \rtimes K) \geq \mathrm{TC}_r(K)$.

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π torsion-free group. For any ascending sequence $\{K_j\}_{j \in I}$ of normal subgroups of π of the form

$$\{1\} = K_0 \leqslant K_1 \leqslant \cdots \leqslant K_i \leqslant K_{i+1} \leqslant \cdots \trianglelefteq \pi$$

there exists a sequence $\{H_j\}_{j \in I}$ of subgroups of $\pi \times \pi$ such that

$$0 \leq \cdots \leq \mathrm{secat}(H_{i+1} \hookrightarrow \pi \times \pi) \leq \mathrm{secat}(H_i \hookrightarrow \pi \times \pi) \leq \cdots \leq \mathrm{TC}(\pi).$$

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$$\underline{\mathrm{genus}}(G) \geq n \qquad \text{but} \qquad \underline{\mathrm{TC}}(G) = \mathrm{TC}(\underline{B}G) = \mathrm{TC}(S^n) = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

¡Gracias por su atención!
Thank you for your attention!
Dziękuję za uwagę!

Talk based on the paper

A. Espinosa Baro *Sectional category of subgroup inclusions and sequential topological complexities of aspherical spaces as $\mathcal{A}$-genus*

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