Sequential topological complexity of aspherical spaces and sectional categories of subgroup inclusions

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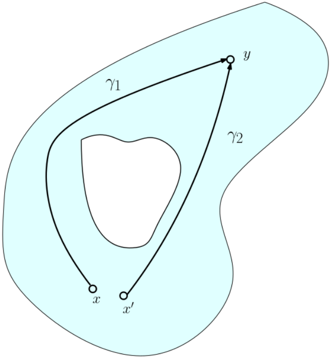

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Topological complexity (Farber '01)

The *topological complexity* of X, $\mathrm{TC}(X)$, is least $k \geq 0$ s.t. there exists an open cover of $X \times X$ by $k+1$ open subsets $\{U_i\}_{0 \leq i \leq k}$ satisfying there exists

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Theorem (Costa, Farber '10)

Let X be a CW-complex with $n = \dim(X) \geq 2$. One has $\mathrm{TC}(X) = 2n$ iff $\mathfrak{v}^{2n} \neq 0$ for a special class

$$\mathfrak{v}^{2n} \in H^{2n}(X \times X; J^{\otimes 2n}) \qquad J = \ker[\mathbb{Z}[\pi_1(X)] \xrightarrow{\varepsilon} \mathbb{Z}]]$$

called canonical class.

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For $\mathrm{ev}_1\colon P_*X \to X$ by $\gamma \mapsto \gamma(1)$ then $\mathrm{secat}(\mathrm{ev}_1) = \mathrm{cat}(X)$. If the fibration is the path space fibration π then $\mathrm{secat}(\pi) = \mathrm{TC}(X)$.

Basic examples

S^{2n+1}:

$U_0 := \{(x,y) | x, y \in S^{2n+1} \text{ with } x \neq -y\}$ $\quad$ $U_1 := \{(x,y) | x, y \in S^{2n+1} \text{ such that } x \neq y\}$.

$s_0(x,y)$ is the shortest geodesic joining x and y. $s_1(x,y)$ is the map which moves x to $-y$ as before, and then $-y$ to y through non-vanishing continuous tangent vector field v

$$-\cos(\pi t)y + \sin(\pi t)\frac{v(y)}{|v(y)|}.$$

S^{2n}: $u \in H^{2n}(S^{2n})$ and define

$$v := u \otimes 1 - 1 \otimes u \in H^{2n}(S^{2n} \times S^{2n}).$$

$\Delta^*(u \otimes 1) = u = \Delta^*(1 \otimes u)$ and $\Delta^*(v) = 0$. Observe

$$\begin{aligned} v \cup v &= ((u \otimes 1) - (1 \otimes u)) \cup ((u \otimes 1) - (1 \otimes u)) \\ &= -(u \otimes 1) \cup (1 \otimes u) - (1 \otimes u) \cup (u \otimes 1) \\ &= -2u \otimes u \neq 0. \end{aligned}$$

By cohomological lower bound we have $\mathrm{TC}(S^{2n}) \geq 2$. By the upper dimensional bound $\mathrm{TC}(S^{2n}) \leq 2$.

Basic examples

$X = \underbrace{S^n \times \cdots \times S^n}_{k}$: We have that

$$\mathrm{TC}(X) \leq \begin{cases} k & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even.} \end{cases}$$

It is an equality. Let $u_i \in H^n(X;\mathbb{Q})$ be the pullback of the fundamental class of S^n via projection onto the i-th factor.

$$\prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1) \neq 0 \text{ if n is odd} \qquad \prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1)^2 \neq 0 \text{ if n is even.}$$

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Σ_g: Cases $g = 1, 2$ seen. So $g \geq 2$. We find 1-dimensional classes $u_1, u_2, v_1, v_2 \in H^1(\Sigma_g, \mathbb{Q})$ satisfying $u_1 u_2 = v_1 v_2 = u_1 v_2 = u_2 v_1 = u_1^2 = u_2^2 = v_1^2 = v_2^2 = 0$ and $u_1 v_1 = u_2 v_2$ is non trivial in $H^2(\Sigma_g, \mathbb{Q})$.

$$\prod_{i=1}^{2}(u_i \otimes 1 - 1 \otimes u_i) \cup (v_i \otimes 1 - 1 \otimes v_i) \neq 0$$

so $\mathrm{TC}(\Sigma_g) \geq 4$. By the dimension connectivity bound $\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4$.

Sequential topological complexities

(Rudyak, '10): For each $r \geq 2$ let

$$p_r \colon PX \to X^r \qquad p_r(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{r-1}\right) \cdots, \gamma\left(\frac{r-2}{r-1}\right), \gamma(1)\right).$$

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- One has $\mathrm{cat}(X^{r-1}) \leq \mathrm{TC}_r(X) \leq \mathrm{cat}(X^r)$.

The Eilenberg-Ganea problem

Recall for G a group $K(G,1)$ is a space with $\pi_1(K(G,1)) = G$ and $\pi_k(K(G,1)) = 0$ $\forall k > 1$. G is geometrically finite if there exists a finite CW model for $K(G,1)$.

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- Farber, Mescher '20: a lower bound on $\mathrm{TC}(G)$ in terms of cohomological dimensions of centralizers.
- Dranishnikov '20: If G is hyperbolic, and $G \ncong \mathbb{Z}$ then $\mathrm{TC}(G) = 2\mathrm{cd}(G)$.
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- Basabe, González, Rudyak, Tamaki '14: $\mathrm{TC}_r(\mathbb{Z}^n) = (r-1)\mathrm{cd}(\mathbb{Z}^n) = (r-1)n$.
- Farber, Oprea '19: generalization of the FGLO bounds.
- Hughes, Li '22: If G is hyperbolic, $G \ncong \mathbb{Z}$, then $\mathrm{TC}_r(G) = r\mathrm{cd}(G)$.

Sectional category of subgroup inclusions

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General bounds are given by

- $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}(G)$.
- $\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{nilker}\left(i^*\colon H^*(G, A) \to H^*(H, \mathrm{Res}^G_H(A))\right)$

$\mathrm{secat}(H \hookrightarrow G)$ and classifying spaces of families

Denote $E_{\langle H \rangle} G$ as the classifying space for the family of subgroups generated by H. By work of Blowers $E_{\langle H \rangle}(G) \simeq *^{\infty}(G/H)$.

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Theorem (Błaszczyk, Carrasquel Vera, EB '20)

$\mathrm{secat}(H \hookrightarrow G)$ *coincides with the minimal* $n \geq 0$ *s.t. the* G*-equivariant map* $\rho \colon EG \to E_{\langle H \rangle} G$ *can be* G*-equivariantly factored up to* G*-homotopy as*

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The map ρ is induced by the universal property of $E_{\langle H\rangle}G$. To prove it we use

Lemma (Błaszczyk, Carrasquel Vera, EB '20)

$\mathrm{secat}(H \hookrightarrow G) \leq n$ iff the Borel fibration

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This generalizes Farber-Grant-Lupton-Oprea '19 and Farber-Oprea '19 for TC and TC^r.

$\mathrm{secat}(H \hookrightarrow G)$ and classifying spaces of families (cont.)

Also, it can be proved:

Theorem

Let G be a torsion-free group, $H \leqslant G$ and $\mathcal{A} = \{G/H\}$.

(a) *For any $\mathcal{F}$ full family of G*

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(G)).$$

(b) *For any subgroup $K \leqslant G$ subconjugate to H such that $\mathrm{cd}_{\langle K \rangle} G \geq 3$ we have*

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle K \rangle} G.$$

(c) *Under (b), if $K \trianglelefteq G$ then*

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Which for TC_r becomes

Corollary

Let π be a torsion-free group, and $K \leqslant \pi^r$ subconjugated to $\Delta_{r,\pi}$.

(a) $\mathrm{TC}_r(\pi) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(\pi^r))$ *for $\mathcal{F}$ any full family of π.*

(b) $\mathrm{TC}_r(\pi) \leq \mathrm{cd}_{\langle K \rangle} \pi^r$.

(c) $\mathrm{TC}_r(\pi) \leq \mathrm{cd}(\pi^r/K)$ *if $K \trianglelefteq \pi^r$.*

The case of normal subgroups

(Grant '12) The cohomological dimension of a group homomorphism $\phi\colon G \to H$, $\mathrm{cd}(\phi)$ is the maximum $k \geq 0$ for which exists H-module A so that

$$\phi^*\colon H^k(H;A) \to H^k(G;\phi^*A)$$

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Let $N \triangleleft G$ and $\pi : G \to Q$ the projection. Then

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To get significant bounds beyond the normal case we will use more elaborated constructions in group cohomology.

The Berstein-Schwarz relative class

For a subgroup $H \leqslant G$ consider the augmentation ideal

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Our main results

Theorem

Let G be a geometrically finite group and $H \leq G$. Let

$$\kappa_{G,H} := \max\{\mathrm{cd}(H \cap xHx^{-1}) \mid x \in G \setminus H\}.$$

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The case $r = 2$ was implicit in Farber-Mescher '20.

Strategy to prove it

The strategy of the proof is to take $\omega \in H^1(G, I)$ and use homological algebra to show that $\omega^{\mathrm{cd}(G)-\kappa_{G,H}} \neq 0$.

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- *There exists $\gamma \in H^{n-k}(G; \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes k}, A))$ with $\alpha = \psi_*(\omega^k \cup \gamma)$, where $\psi \colon I^{\otimes k} \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes k}, A) \to A$ is $\psi(x_1 \otimes \cdots \otimes x_k \otimes f) = f(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1)$.*

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- *There exists $\gamma \in H^{n-k}(G; \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes k}, A))$ with $\alpha = \psi_*(\omega^k \cup \gamma)$, where $\psi\colon I^{\otimes k} \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes k}, A) \to A$ is $\psi(x_1 \otimes \cdots \otimes x_k \otimes f) = f(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1)$.*
- *α lies in the image*

$$\delta^{n-1,1} \circ \delta^{n-2,2} \circ \cdots \circ \delta^{n-k,k} \colon \mathrm{Ext}^{n-k}_{\mathbb{Z}[G]}(I^{\otimes k}, A) \to \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong H^n(G, A).$$

Corollary

If $\mathrm{Im}\left[\delta^{n-1,1} \circ \delta^{n-2,2} \circ \cdots \circ \delta^{n-k,k} \colon \mathrm{Ext}^{n-k}_{\mathbb{Z}[G]}(I^{\otimes k}, A) \to H^n(G, A)\right] \neq 0$ *then*

$$\mathrm{secat}(H \hookrightarrow G) \geq k.$$

Forming a spectral sequence

To show that image is non-zero, we can assemble our Ext-sequences into an exact couple

$$\begin{array}{ccc} D_0 & \xrightarrow{\;i_0\;} & D_0. \\ & {\scriptstyle k_0}\nwarrow \quad \swarrow{\scriptstyle j_0} & \\ & E_0 & \end{array}$$

$$D_0^{r,s} := \mathrm{Ext}^r_{\mathbb{Z}[G]}(I^{\otimes s}, A) \qquad E_0^{r,s} := \mathrm{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^{\otimes s}, A) \qquad i_0 := \sum_{r,s \in \mathbb{N}_0} \delta^{r,s}.$$

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- Let $s \in \{0, 1, \dots, n-1\}$. Then $u \in D_{s+1}^{n,0}$ if and only if

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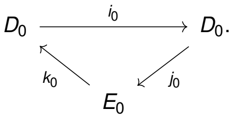

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In these terms, the corollary turns to be

Corollary

Let $n, p \in \mathbb{N}$ with $p \leq n$. If $D_p^{n,0} \neq \{0\}$, then $\omega^p \neq 0$ and thus $\mathrm{secat}(H \hookrightarrow G) \geq p$.

The sketch of the proof

Consider the left H-action on G/H by $H \times G/H \to G/H$, $\quad h \cdot gH = (hg)H$.

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Theorem

For each $C \in \mathcal{C}'_s(G/H)$ fix a representative $x_C \in C$ and let $N_C := H_{x_C}$ be the isotropy group of x_C. Then

$$E_0^{r,s} \cong \prod_{C \in \mathcal{C}'_s(G/H)} H^r(N_C; \mathrm{Res}^G_{N_C}(A)) \quad \forall r \in \mathbb{N}$$

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For $u \in D_0^{n,0}$ one identifies obstructions to $u \in D_k^{n,0}$ lying in the groups

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In particular, $H^r(N_C; \mathrm{Res}_{N_C}^G(A)) = 0$ whenever $r > \kappa_{G,H}$, so we derive from Theorem that

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Put $d := \mathrm{cd}(G)$. If there is A such that $H^d(G, A) \neq 0$ we derive from

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that $D^{d,0}_{d-\kappa_{G,H}} \neq 0$ Then by the corollary, $\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{cd}(G) - \kappa_{G,H}$.

An example of application

Let G be a group. A subgroup $H \leq G$ is *malnormal* if

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Corollary

Let π_1 and π_2 be geometrically finite groups and consider a free product with amalgamation $\pi_1 *_H \pi_2$, such that H is malnormal in π_1 or malnormal in π_2. Then for each $r \geq 2$

$$\mathrm{TC}_r(\pi_1 *_H \pi_2) \geq r \cdot \mathrm{cd}(\pi_1 *_H \pi_2) - \max\{k(\pi_1), k(\pi_2)\}.$$

Parametrized TC of group epimorphisms

For fibration $p : E \to B$ (with fibre X) set
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Theorem

Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] \geq \mathrm{cd}(G \times_Q G) - k(\rho),$$

where $k(\rho) = \max\{\mathrm{cd}(C(g)) \mid g \in \ker\rho,\ g \neq 1\}$.

Canonical class for non-aspherical spaces

Let $\pi = \pi_1(X)$ $\qquad I_r := \ker\left[\varepsilon : \mathbb{Z}[\pi^{r-1}] \to \mathbb{Z}\right]$. Define

$$f_r : \pi^r \to I_r, \quad f_r(g_1, g_2, \ldots, g_r) = (g_1 g_2^{-1} - 1, g_2 g_3^{-1} - 1, \ldots, g_{r-1} g_r^{-1} - 1)$$

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There is an obstruction to a continuous section of p_r over the 1-skeleton of X^r

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Corollary

X connected CW complex with $\pi_1(X) = \pi$, let $K = K(\pi, 1)$ and $f_X : X \to K$ a classifying map for the universal cover of X. Then

$$\mathfrak{v}_r^X = (f_X^r)^*(\mathfrak{v}_r^K) \in H^1(X^r, I_r).$$

Auxiliar tool: the diagonal topological complexity

(Farber, Oprea '19) The *r-th D-topological complexity*, $\mathrm{TC}_r^{\mathrm{D}}(X)$, is the minimal $k \geq 0$ s.t.

$$X^r = U_0 \cup U_1 \cup \dots U_k$$

with U_i open, s.t. for each choice of base point $u_i \in U_i$ the map $\pi_1(U_i, u_i) \xrightarrow{i_*} \pi_1(X^r, u_i)$ takes values in a subgroup of π^r conjugated to $\Delta_{\pi,r}$.

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Theorem (Farber, Oprea '19)

Let $K = K(\pi, 1)$ connected and finite, and let $q : \widehat{K^r} \to K^r$ be the connected covering space of $\Delta_r \subset \pi^r$. Then

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a) $\mathrm{TC}_r(X) \geq \mathrm{TC}_r^{\mathcal{D}}(X)$.

b) *Let $q^X : \widehat{X}^r \to X^r$ be the covering of X^r associated with $\Delta_r \subset \pi^r$. Then*

$$\mathrm{TC}_r^{\mathcal{D}}(X) = \mathrm{secat}(q^X : \widehat{X}^r \to X^r).$$

Lower bounds for non-aspherical spaces

Lemma

Let X be a connected CW-complex s.t. $\widetilde{X}$ is $(k-1)$-connected, and put $\pi := \pi_1(X)$. If $\mathrm{cd}(\pi) \leq k$, then $\mathrm{TC}_r^{\mathrm{D}}(X) = \mathrm{TC}_r(\pi)$.

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Under strong enough hypothesis, we can generalize our lower bound to non-aspherical spaces:

Theorem

Let π be a geometrically finite group and X be a connected locally finite CW-complex with $\pi_1(X) = \pi$ and such that $\widetilde{X}$ is $(k-1)$-connected. If $\mathrm{cd}(\pi) \leq k$, then

$$\mathrm{TC}_r(X) \geq r \cdot \mathrm{cd}(\pi) - k(\pi).$$

The height of the canonical class for non-aspherical spaces

$$p^X : \widetilde{X^r} \times_G \pi^{r-1} \to X^r \equiv q^X : \widehat{X}^r \to X^r \qquad \mathrm{TC}_r^{\mathrm{D}}(X) = \mathrm{secat}\big(p^X : \widetilde{X}^r \times_G \pi^{r-1} \to X^r\big).$$

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The $(k+1)$-join of p^X is given by

$$p^X_k : \widetilde{X}^r \times_G E_k(\pi^{r-1}) \to X^r, \qquad E_k(\pi^{r-1}) = \pi^{r-1} * \pi^{r-1} * \cdots * \pi^{r-1}.$$

Thus $\mathrm{TC}^{\mathrm{D}}_r(X) = \inf\{k \in \mathbb{N} \mid q_k : \widetilde{X}^r \times_G E_k(\pi^{r-1}) \to X^r$ admits a continuous section$\}$.

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Let $K = K(\pi, 1)$ and $f_X^r : X^r \to K^r$ the classifying map of the universal cover of X^r.
Consider the pullback diagram for any $k \in \mathbb{N}$:

$$\begin{array}{ccc} \widetilde{X}^r \times_G E_k(G/\Delta) & \longrightarrow & \widetilde{K}^r \times_G E_k(G/\Delta) \\ {\scriptstyle q_k^X}\downarrow & & \downarrow{\scriptstyle q_k^K} \\ X^r & \xrightarrow{\quad f_X^r \quad} & K^r. \end{array}$$

Combining this with previous results we can generalize properties of canonical classes to cell complexes that are not necessarily aspherical.

Theorem

Let X be a connected finite cell complex. Then $\mathrm{TC}_r^{\mathrm{D}}(X) \geq \mathrm{height}(\mathfrak{v}_r)$.

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Corollary

Suppose X is a connected n-dimensional CW complex and that the universal cover $\widetilde{X}$ is $(rn-k)$-connected with $\mathrm{TC}_r^{\mathrm{D}}(X) \geq k$ and $\mathrm{cd}(\pi) \leq rn-k+1$. Then

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Suppose X is a connected n-dimensional CW complex and $\mathrm{TC}_r(X) = rn$. Then $\mathrm{TC}_r^{\mathrm{D}}(X) = rn$ as well.

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We conclude with a generalization of the Costa-Farber theorem:

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$$H^i(X^r; \pi_{i-1}(E_{rn-1}(\pi))) \qquad 1 \leq i \leq rn.$$

$E_{rn-1}(\pi)$ is $(rn-2)$-connected space, so all but the primary obstruction vanish.

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Corollary

Let X be a connected n-dimensional finite CW complex, where $n \in \mathbb{N}$, whose fundamental group is free abelian of rank at most n. Then $\mathrm{TC}_r(X) < rn$.

¡Gracias por su atención!
Thank you for your attention!
Dziękuję za uwagę!

Talk (mostly) based on the paper

A. Espinosa Baro, M. Farber, S. Mescher, J. Oprea, *Sequential topological complexity of aspherical spaces and sectional categories of subgroup inclusions*, published (online) in Mathematische Annalen
https://doi.org/10.1007/s00208-024-03033-1

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NARODOWE CENTRUM NAUKI

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https://sites.google.com/view/applied-topology-2025

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