Topics on topological robotics
On topological complexity of Eilenberg-MacLane spaces and effective topological complexity

Arturo Espinosa Baro
Supervised by prof. dr. hab. Wacław Bolesław Marzantowicz
Associate advisor Zbigniew Błaszczyk

Adam Mickiewicz University
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Adam Mickiewicz University in Poznań

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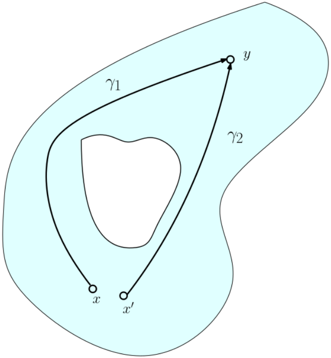

A topological feature of the configuration space inducing instability on the motion planning.

Topological complexity and sectional category

Topological complexity (Farber '01) $\mathrm{TC}(X) = \min k$ s.t. $\exists \{U_i\}_{0 \le i \le k}$ open cover of X with (cont.) local sections of $\pi \colon PX \to X \times X$.

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As secat is homotopy invariant, $\mathrm{TC}_r(X) = \mathrm{secat}(\Delta_r\colon X \hookrightarrow X^r)$.

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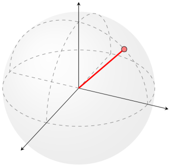

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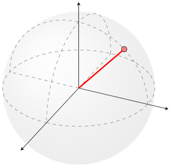

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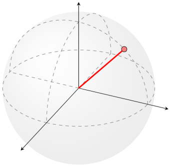

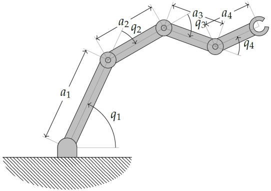

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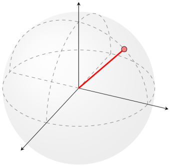

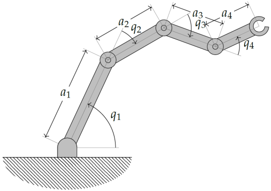

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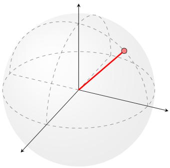

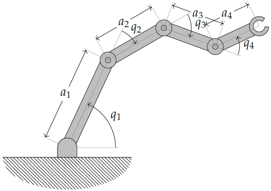

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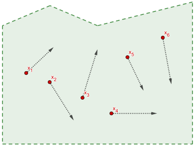

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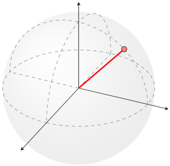

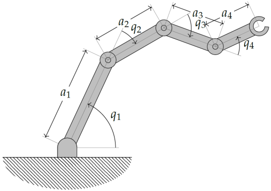

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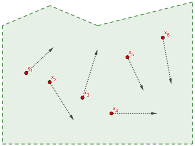

$$\mathrm{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n-2 & \text{for all } m \text{ odd} \\ 2n-3 & \text{for all } m \text{ even} \end{cases}$$

(Farber-Yuzvinsky '04, Farber-Grant '08)

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An interesting homotopy invariant connected with classic invariants (LS-cat, secat...) with its own open problems like the Eilenberg-Ganea problem.

The Eilenberg-Ganea problem

$$K(G,1) \qquad \begin{cases} \pi_1(K(G,1)) = G \\ \pi_k(K(G,1)) = 0 \end{cases}.$$

G is geometrically finite if $\exists$ a finite CW model for $K(G,1)$.

Define the (sequential) topological complexities of a group by

$$\mathrm{TC}_r(G) = \mathrm{TC}_r(K(G,1)).$$

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The problem remains open.

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- Dranishnikov '17, Cohen-Vandembroucq '17: N closed non-orientable surface, $N \neq \mathbb{R}P^2 \Rightarrow \mathrm{TC}(N) = 4$.
- Farber-Mescher '20: lower bound by dimensions of centralizers.
- Dranishnikov '20: G hyperbolic, $G \ncong \mathbb{Z} \Rightarrow \mathrm{TC}(G) = 2\mathrm{cd}(G)$.
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For $\mathrm{TC}_r(G)$, $r \geq 2$: not so much is known.

- Basabe-González-Rudyak-Tamaki '14: $\mathrm{TC}_r(\mathbb{Z}^n) = (r-1)\mathrm{cd}(\mathbb{Z}^n) = (r-1)n$.
- Farber-Oprea '19: generalize FGLO bounds.
- Hughes-Li '22: G hyperbolic, $G \ncong \mathbb{Z} \Rightarrow \mathrm{TC}_r(G) = r\mathrm{cd}(G)$.

Sectional category of subgroup inclusions

For $\iota\colon H \hookrightarrow G$ define the sectional category of the monomorphism ι by

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Theorem (EB, Farber, Mescher, Oprea)

Let $N \lhd G$ and $\pi : G \to Q = G/N$ the projection. Then

$$\mathrm{cd}(\pi : G \to Q) \leq \mathrm{secat}(N \hookrightarrow G) \leq \mathrm{cd}(Q).$$

If $\pi^* : H^{\mathrm{cd}(Q)}(Q, A) \to H^{\mathrm{cd}(Q)}(G, \pi^* A)$ is non-zero then $\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(Q)$.

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To go further, we use some homological algebra! But first just a bit of equiv. homotopy...

$\mathrm{secat}(H \hookrightarrow G)$ and classifying spaces of families

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Theorem (Błaszczyk, Carrasquel, EB)

secat$(H \hookrightarrow G)$ coincides with min. $n \geq 0$ s.t. $\rho \colon EG \to (E_{\langle H \rangle} G)_n$ can be factorized up to G-homotopy as

$$\begin{array}{ccc} EG & \xrightarrow{\quad\rho\quad} & E_{\langle H \rangle} G \\ & \searrow \; \nearrow & \\ & (E_{\langle H \rangle} G)_n, & \end{array}$$

We also introduce Adamson cohomology in the investigation.

secat and TC$_r$ as $\mathcal{A}$-genus

G group, X a G-space, $\mathcal{A}$ family of G-spaces.

$$\mathcal{A}\text{-genus}(X) := \min\{k \in \mathbb{Z}^+ \mid \exists \{U_i\}_{0 \le i \le k}, U_i \subset X \text{ open s.t. } \forall 0 \le i \le k \ \exists A_i \in \mathcal{A} \text{ and } G\text{-equiv } U_i \to A_i\}$$

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Theorem (EB)

Let X be a path conn. CW-complex with $\pi_1(X) = \pi$. If $q : \widehat{X} \to X$ is a conn. covering:

$$\mathrm{secat}(q) = \mathcal{A}\text{-genus}(\widetilde{X}) \qquad \mathcal{A} = \left\{ \pi \big/ \pi_1(\widehat{X}) \right\}.$$

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Corollary (EB)

Let π be a torsion-free group, and $K \leqslant \pi^r$ subconj. to $\Delta_{r,\pi}$.

(a) $\mathrm{TC}_r(\pi) \le \mathcal{A}\text{-genus}(E_{\mathcal{F}}(\pi^r))$ *for $\mathcal{F}$ any full family of π.*

(b) $\mathrm{TC}_r(\pi) \le \mathrm{cd}_{\langle K\rangle} \pi^r$.

(c) $\mathrm{TC}_r(\pi) \le \mathrm{cd}(\pi^r / K)$ *if $K \trianglelefteq \pi^r$.*

The Berstein-Schwarz relative class

For $H \leqslant G$ we define the relative augmentation ideal

$$\sigma\colon \mathbb{Z}[G/H] \to \mathbb{Z} \qquad \sigma\left(\sum_{x\in G/H} n_x \cdot x\right) = \sum_{x\in G/H} n_x \qquad I := \ker\sigma$$

The Berstein-Schwarz class of G relative to H is $\omega \in H^1(G, I)$ represented by

$$\xi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\otimes K, I), \qquad \xi = \mu\circ(\varepsilon\otimes \mathrm{id}_K) \qquad K \text{ augm. ideal} \qquad \mu\colon K \to I$$

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- $\omega \in \ker[H^1(G, I) \xrightarrow{i^*} H^1(H, I)]$.
- *If* $\omega^k \neq 0 \Longrightarrow \mathrm{secat}(H \hookrightarrow G) \geq k$.
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A class $\alpha \in H^n(G, A)$, $\alpha \neq 0$ is essential relative to H if $\exists \varphi\colon I^n \to A$ s.t. $\varphi_*(\omega^n) = \alpha$.

Lower bounds for secat and TC_r

A spectral sequence captures conditions for a class to be essential.

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From $K(G, 1)$ to symmetries

Now we look at TC not of $K(G, 1)$ but of G-spaces (symmetries)!

Motion planning and symmetries

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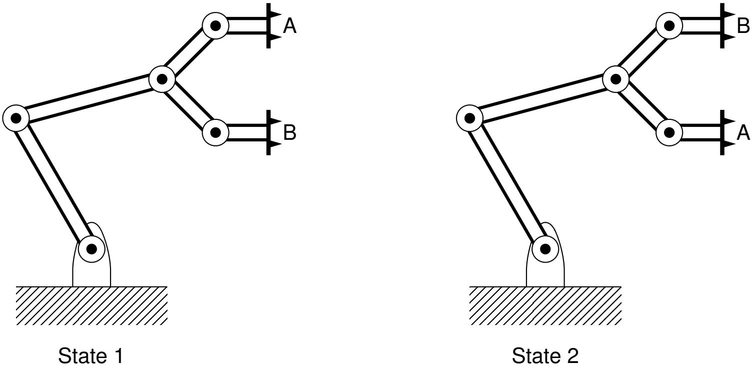

Two different but functionally equivalent states of the robot arm.

Effective TC and cat

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If $G \to P \to B$ principal G-bundle, $\mathrm{TC}^{G,\infty}(P) \leq \mathrm{TC}(B) \leq 2(\dim(P) - \dim(G))$.

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Furthermore, if $|G| \leq \mathrm{cd}(X)$, then

$$TC^{G,2}(X) > 0.$$

¡Gracias por su atención!
Thank you for your attention!
Dziękuję za uwagę!

Papers part of this dissertation:

- *"On the sectional category of subgroup inclusions and Adamson cohomology theory"*. Joint with Z. Błaszczyk and J. G. Carrasquel-Vera. Published in J. Pure and Appl. Alg. vol. 226, issue 6, June 2022, 106959.
- *Sectional category and sequential topological complexity of aspherical spaces as* $\mathcal{A}$-genus. Preprint.
- *On properties of effective topological complexity and effective Lusternik-Schnirelmann category.* Joint with Z. Błaszczyk and A. Viruel. To appear in Proc. R. Soc. Edinb., Sect. A, Math.
- *Sequential topological complexity of aspherical spaces and sectional category of subgroup inclusions.* Joint with M. Farber, S. Mescher and J. Oprea. Published in Math. Ann. (online).

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