A cohomological characterization of nilpotent fusion systems

Arturo Espinosa Baro

Joint work with Antonio Diaz Ramos and Antonio Viruel Arbaizar

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2. In the 90's, the Martino Priddy conjecture arrives: For any prime p and any pair G_1, G_2 of finite groups, $BG_{1_p}^{\wedge} \simeq BG_{2_p}^{\wedge}$ if and only if there is a fusion preserving isomorphism of Sylow p-subgroups $S_1 \xrightarrow{\cong} S_2$. That is, the homotopy type of the p-completion of the group is codified by its fusion system. This brings homotopy theory directly into play.

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3. In 2001, Broto, Levi and Oliver introduced the notion of *p*-local finite group (a fusion system with an associated classifying space). Later, in 2013, Chermak and Oliver showed that every fusion system admits such classifying space.

The setting: saturated fusion systems

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An abstract fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following two conditions:

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Fully normalized and centralized, and receptive subgroups

- $P \leq S$ is fully centralized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all P' which is $\mathcal{F}$-conjugated to P.

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A subgroup $P \leq S$ is receptive if, for every $Q \leq S$ $\mathcal{F}$-conjugate to P and every $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set

$$N_\varphi = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}_{|_Q} = \varphi$.

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Saturated fusion systems

$\mathcal{F}$ is a saturated fusion system if each subgroup $P \leq S$ is $\mathcal{F}$-conjugate to at least one subgroup which is fully normalized and receptive.

Nilpotency for fusion systems

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If G is a finite group, G is p-nilpotent if and only if the induced p-fusion system $\mathcal{F}_G(S)$ is nilpotent.

p-local finite groups and classifying spaces of fusion systems

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$\mathcal{F}$-centric subgroups

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The transporter category

The transporter category $\mathcal{T}_S(G)$ for $S \leq G$ is a category whose objects are subgroups of S and whose morphism sets are:

$$\mathrm{Mor}_{\mathcal{T}_S(G)}(P, Q) = T_G(P, Q) = \{g \in G | gPg^{-1} \in Q\}$$

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2. We have that δ is the identity on objects, and π is the inclusion on objects. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ such that P is fully centralized in $\mathcal{F}$, $C_S(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ and

   $$\pi_{P,Q} : \mathrm{Mor}_{\mathcal{L}}(P, Q) \to \mathrm{Hom}_{\mathcal{F}}(P, Q)$$

   is the orbit map for this action.

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3. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and each $g \in T_S(P, Q)$, $\pi_{P,Q}$ sends $\delta_{P,Q}(g) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ to $c_g \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

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4. For all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and all $g \in P$,
$$\delta_{Q,Q}(\pi(\psi)(g))\psi = \psi\delta_{P,P}(g)$$
commutes in $\mathcal{L}$.

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Centric linking systems

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p-local finite groups

A finite p-local group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group, $\mathcal{F}$ is a fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$

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Classic nilpotency criteria for groups

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Let $\mathcal{H}$ denote Tate's cohomology. Then, we have the following classic result:

Wong, Hoechsmann+Roquette+Zassenhaus, 1968

Let G be a finite group. Then, the following are equivalent:

G is p-nilpotent.

For every finitely generated $\mathbb{F}_p[G]$-module M, if $\mathcal{H}^k(G, M) = 0$ for some k, then $\mathcal{H}^n(G, M) = 0$ for every n (i.e. M is $\mathcal{H}$-acyclic).

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Then $\mathcal{H}^n(G, M) \simeq \mathcal{H}^{n+1}(G, B) \simeq \mathcal{H}^{n-1}(G, A)$, and we proceed arguing on dimensions 0 and 1.

Modules over a fusion system

$\mathbb{F}_p[\mathcal{F}]$-modules

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Every $\mathbb{F}_p[\mathcal{F}]$-module is a $\mathbb{F}_p[S/\mathfrak{hyp}(\mathcal{F})]$-module, where

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$$\mathfrak{hyp}(\mathcal{F}) = \langle [P, O^p(Aut_{\mathcal{F}}(P))],\ P \leq S \rangle$$

Cohomology of saturated fusion systems

Let $\mathcal{F}$ be a saturated fusion system over S, and M a $\mathbb{F}_p[\mathcal{F}]$-module. For each $n \geq 0$ define the twisted cohomology group $\mathrm{H}^n(\mathcal{F}^c, M)$ as the $\mathcal{F}^c$-stable elements:

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$$\mathrm{H}^n(\mathcal{F}^c, M) = \{z \in \mathrm{H}^n(S, M) | \forall P \in \mathcal{F}^c, \, \forall \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S) : \, res(z) = \varphi^*(z)\}$$

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$$\mathrm{H}^*(\mathcal{F}^c, M) \simeq \mathrm{H}^*(B\mathcal{F}, M)$$

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Theorem

Let $\mathcal{F}$ be a fusion system. Then, the following are equivalent:

- $\mathcal{F}$ is nilpotent
- For each $\mathbb{F}_p[\mathcal{F}]$-module M, if $\mathrm{H}^m(\mathcal{F}^c, M) = 0$ for some $m > 0$, then $\mathrm{H}^n(\mathcal{F}^c, M) = 0$ for every $n > 0$.

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For (1) $\Rightarrow$ (2) we can reduce to the group theoretical proof.

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For (2) $\Rightarrow$ (1), it's time to use topology. Essentially, we set $\pi = S/\mathfrak{hyp}(\mathcal{F})$, and consider the universal covering space

$$\pi \to BO^p(\mathcal{F}) \to B(\mathcal{F})$$

Therefore, if $M = \mathbb{F}_p[\pi]$, with π acting by left multiplication, then

$$\mathrm{H}^1(B\mathcal{F}; M) \cong \mathrm{H}^1(BO^p(\mathcal{F}); \mathbb{F}_p) = 0$$

as $BO^p(\mathcal{F})$ is simply connected. By hypothesis

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for all $n \geq 1$. As $BO^p(\mathcal{F})$ is a p-complete space, then it must be contractible, so $O^p(\mathcal{F})$ is the trivial system,

A cohomological characterization of nilpotent fusion systems

Theorem

Let $\mathcal{F}$ be a fusion system. Then, the following are equivalent:

- $\mathcal{F}$ is nilpotent
- For each $\mathbb{F}_p[\mathcal{F}]$-module M, if $\mathrm{H}^m(\mathcal{F}^c, M) = 0$ for some $m > 0$, then $\mathrm{H}^n(\mathcal{F}^c, M) = 0$ for every $n > 0$.

For (2) $\Rightarrow$ (1), it's time to use topology. Essentially, we set $\pi = S/\mathfrak{hyp}(\mathcal{F})$, and consider the universal covering space

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Therefore, if $M = \mathbb{F}_p[\pi]$, with π acting by left multiplication, then

$$\mathrm{H}^1(B\mathcal{F}; M) \cong \mathrm{H}^1(BO^p(\mathcal{F}); \mathbb{F}_p) = 0$$

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$$0 = \mathrm{H}^n(B\mathcal{F}; M) \cong \mathrm{H}^n(BO^p(\mathcal{F}); \mathbb{F}_p)$$

for all $n \geq 1$. As $BO^p(\mathcal{F})$ is a p-complete space, then it must be contractible, so $O^p(\mathcal{F})$ is the trivial system, hence $\pi = S$, and $\mathcal{F}$ is nilpotent.

Muchas gracias!